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THE BOUNDARY VALUE PROBLEM OF THE THEORY OF VISCOELASTIC PLASTICITY OF A GROWING BODY SUBJECT TO AGING*

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The formulation of the boundary value problem for a visco-elastic-plastic growing body taking into account its increasing inhomogeneity caused by the sequence and speed of accretion is considered. The presence of two changing boundaries, namely, the external surface of the body and the boundary of the material plastic state, is characteristic for this formulation. The complete system of equations and conditions of the boundary value problem for a continuously growing body is presented. As an example of the application, a model problem is given of a non-uniformly aging hollow sphere with a shifting outer surface. Conclusions are drawn regarding the substantial effect of the accretion speed and the preliminary stress of the growing elements on the stress-strain state of the body.

Problems of the accretion of a deformed solid are encountered when investigating technological processes such as winding, spraying, and icing, and also in the gradual erection of structures and the growth of bodies of organic and inorganic origin. Similar processes occur in phase transformations in materials, in the polymerization and crystallization in amorphous bodies, etc.

Since actual accretion processes usually take a considerable time, the effect of aging and creep of the material becomes important when determining the stress-strain state of growing bodies. Certain model problems of the accretion of viscoelastic non-uniformly aging bodies were investigated in /1,2/.

To estimate the strength and load-carrying ability of growing bodies it is, however, necessary to take into account that at fairly high stress levels, regions of plastic state may appear and develop in the material. Because various elements of the body during growth originate (are produced) at different instants of time, its plasticity limit k at the point $\mathbf{x} = \{x_1, x_2, x_3\}$ together with other physicomechanical properties depend on the material age at the point, i.e. are a function of time and the space coordinates $k = k(t - \tau^*(\mathbf{x}))$, where $\tau^*(\mathbf{x})$ is the instant of appearance of the element of the body whose position in Cartesian coordinates is defined by the vector $\mathbf{x}$. Various aspects of the theory of plasticity of inhomogeneous bodies and a review of publications in this area can be found in /3/. General problems of the mechanical behaviour of viscoelastic-plastic bodies were considered in /4/.

1. A model of an inhomogeneously aging viscoelastic-plastic body. The defining equations. Let us assume that the region Ω occupied by the body is divided by the surface S_{12} into a region Ω_1 where the material is in a viscoelastic state, and a region Ω_2 of plastic state. We denote the instant when the material at the point $\mathbf{x}$ transfers into the plastic state by $\tau^+(\mathbf{x})$. We assume that at $t \leqslant \tau^+(\mathbf{x})$ the total shear deformation at the point considered is the sum of the instantaneous elastic and of viscous components $e_{ij} = e_{ij}^E + e_{ij}^V$, while at $t \geqslant \tau^+(\mathbf{x})$ it is equal to $e_{ij} = e_{ij}^E + e_{ij}^V + e_{ij}^P$, where e_{ij}^P are the components of the plastic deformation deviator (here and everywhere below $i, j = 1, 2, 3$).

We will take for the total shear deformation in the viscoelastic region the defining equation of the linear theory of viscoelasticity of inhomogeneously aging media /5/

$$e_{ij}(t, \mathbf{x}) = \mathbf{L}(s_{ij}), \ t \geqslant \tau^*(\mathbf{x}), \ \mathbf{x} \in \Omega_1 \tag{1.1}$$

where s_{ij} are the components of the stress deviator and $\mathbf{L}$ is the linear integral opeartor acting on some function $\alpha(t, \mathbf{x})$ by the rule

$$\mathbf{L}(\alpha) = \frac{\alpha(\mathbf{x})}{2G(t - \tau^*(\mathbf{x}))} - \int_{\tau^*(\mathbf{x})}^{t} \frac{\alpha(\tau, \mathbf{x})}{2G(\tau - \tau^*(\mathbf{x}))} Q(t - \tau^*(\mathbf{x}), \tau - \tau^*(\mathbf{x}))\, d\tau$$

where $G(t)$ is the time-dependent instantaneous elastic shear modulus, and $Q(t, \tau)$ is the creep kernel, determined from creep experiments on uniaxial stretching.

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We will write the condition of plasticity and its corresponding law of flow (the dot at the top denotes the partial derivative with respect to)

$$f(\sigma_{ij})=0, \quad e_{ij}^{P\cdot}=\lambda\frac{\partial f}{\partial\sigma_{ij}}, \quad \mathbf{x}\in\Omega_2 \tag{1.2}$$

where σ_{ij} are the components of the stress tensor, and λ is some positive coefficient of proportionality when $f^{\cdot}(\sigma_{ij})=0$. The case when $f^{\cdot}(\sigma_{ij})<0$, which corresponds to unloading, is not considered here. Taking the Mises plasticity condition, we have in the plastic region the following defining relations:

$$s_*(t,\ \mathbf{x})=k\,(t-\tau^*(\mathbf{x})), \quad s_*=({}^1/_2 s_{ij}s_{ij})^{1/2} \tag{1.3}$$

$$e_{ij}^{P\cdot}(t,\mathbf{x})=\frac{e_*^{P\cdot}(t,\mathbf{x})}{k\,(t-\tau^*(\mathbf{x}))}s_{ij}(t,\mathbf{x}) \tag{1.4}$$

$$e_*^{P\cdot}=({}^1/_2 e_{ij}^{P\cdot}e_{ij}^{P\cdot})^{1/2}, \quad \mathbf{x}\in\Omega_2$$

where s_*, $e_*^{P\cdot}$ are the stress deviator intensities and plastic deformation rates, respectively (the dot in the notation $e_*^{P\cdot}$ has a symbolic meaning).

The volume deformation is, for simplicity, assumed to be ideally elastic and independent of the stress level

$$\varepsilon_{mm}(t,\mathbf{x})=\frac{\sigma_{mm}(t,\mathbf{x})}{3K\,(t-\tau^*(\mathbf{x}))}, \quad m=1,2,3, \quad t\geqslant\tau^*(\mathbf{x}),\mathbf{x}\in\Omega \tag{1.5}$$

where ε_{ij} are the components of the total deformation tensor, and K is the variable modulus of elastic volume deformation.

2. Statement and basic equations of the boundary value problem of the theory of viscoelastic-plasticity for a growing body. In the region occupied by the body the equations of quasistatic equilibrium must be satisfied (inertia effects during accretion are neglected) and the Cauchy geometric relations, which for a body in the process of accretion are expressed in terms of velocities /5/

$$\sigma_{ij,j}+X_i=0, \quad t\geqslant\tau^*(\mathbf{x}), \quad \mathbf{x}\in\Omega \tag{2.1}$$

$$\varepsilon_{ij}^{\cdot}=\frac{1}{2}(u_{i,j}^{\cdot}+u_{j,i}^{\cdot}), \quad t\geqslant\tau^*(\mathbf{x}), \quad \mathbf{x}\in\Omega \tag{2.2}$$

where X_i, u_i are vectors of the volume forces and displacements, respectively. The quantities that determine the formulation (and consequently the solution) of the problem obviously depend on time and the space coordinates. For simplicity, the arguments of these quantities are sometimes omitted.

Since in formulas (2.2) we have the first derivatives with respect to, the formulation of the boundary value problem for the growing body must be supplemented by the initial values of the stress and deformation tensor components /6, 7/ (in this connection the boundary value problem considered here could be naturally called the "initial" boundary value problem). This requirement as applied to the conditions of continuous growth is a reflection of the physically obvious fact that a generated element before joining the main body is in no way connected with the latter. For the theoretical possibility of determining the stress-strain state of the composite body obtained, it is necessary to specify the complete stress state of the element being deposited (the state of the body prior to the deposition of recurrent elements is assumed known).

Suppose that at the instant the element is generated, coincident with the instant of its accretion, its stresses are given by the tensor

$$\sigma_{ij}^{\circ}(\mathbf{x})=\sigma_{ij}(\tau^*(\mathbf{x}),\mathbf{x}) \tag{2.3}$$

The magnitude of the preliminary (initial) deformation of the element $\varepsilon_{ij}^{\circ}(\mathbf{x})=\varepsilon_{ij}(\tau^*(\mathbf{x}),\mathbf{x})$ in this case is determined from the respective defining equations. We shall consider the case when the preliminary stresses do not exceed the initial plastic limit. Then in the model of the medium considered the relation between the prestress tensor and that of initial deformation is defined by the instantaneous elastic part of the equation of state (1.1) and Eq.(1.5) when $t=\tau^*(\mathbf{x})$

$$\varepsilon_{ij}^{\circ}(\mathbf{x})=\frac{\sigma_{ij}^{\circ}(\mathbf{x})}{2G(0)}+\frac{1}{3}\delta_{ij}\left[\frac{1}{3K(0)}-\frac{1}{2G(0)}\right]\sigma_{mm}^{\circ}(\mathbf{x}) \tag{2.4}$$

where δ_{ij} is the Kroneker delta.

Equation (2.2) implies that the initial displacement vector $u_i^{\circ}(\mathbf{x})=u_i(\tau^*(\mathbf{x}),\mathbf{x})$ is immaterial for the stress-strain state of the growing body, and can be arbitrarily selected from some supplementary considerations.

The boundary conditions of the boundary value problem have the form

$$\sigma_{ij}n_j = p_i, \quad t \geqslant \tau^* (\mathbf{x}), \quad \mathbf{x} \in S_\sigma \tag{2.5}$$

$$u_i = v_i, \quad t \geqslant \tau^* (\mathbf{x}), \quad \mathbf{x} \in S_u \tag{2.6}$$

where S_σ, S_u are the parts of the body surface on which the stress vectors p_i and displacement v_i, respectively are specified, and n_i is the unit vector of the normal to the body surface. The symbol $S_* (t)$ denotes that part of body surface over which, at the instant of time t, its accretion takes place. Fig.1 shows schematically the boundaries that divide the body into sections with different boundary conditions. The dashed lines denote the positions of the moving section of the external boundary S_* at separate instants of time, and that of the changing boundary S_{12} $(t_1 < t_2 < t_3)$ of the plastic state of the material.

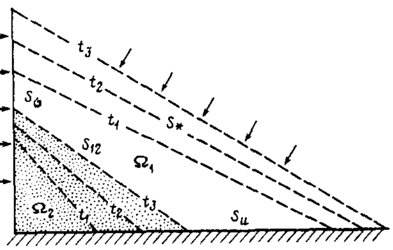

Fig.1

The stationary sections of the external boundaries S_σ and S_u in the course of accretion of the body, generally speaking, expand. The stress vector that determines the action of the external force on the surface of growth can be specified on S_*. The initial values of the components of the stress tensor $\sigma_{ij}{}^\circ (\mathbf{x})$ must be specified taking this action into account.

In fact, conditions (2.5), when they are specified on a part of the growing surface must be regarded as three relations imposed on the initial values of the six stress tensor components. In this case this means that only three components of the stress tensor can be specified independently on the growing surface, namely, the three components of the tensor $\sigma_{ij}{}^\circ$ acting on a small area with normal n_i are determined by the components of the specified external stresses vector p_i from (2.5). The remaining three components of the tensor $\sigma_{ij}{}^\circ$ are specified independently, and represent the controllable initial stress of the accreted element /6/.

We will assume for simplicity that the set of six functions $\sigma_{ij}{}^\circ$ and the function τ^* are continuous with respect to the space coordinates, and the functions k, G, K are continuous with respect to time.

On the separation surface $S_{12} (t)$ of the viscoelastic and plastic regions we stipulate that the conditions of continuity of all the stress tensor components and of the displacement vector are satisfied, namely,

$$\begin{gathered} \sigma_{ij}{}^{(1)} (t, \ \mathbf{x}) = \sigma_{ij}{}^{(2)} (t, \ \mathbf{x}) \\ u_i{}^{(1)} (t, \ \mathbf{x}) = u_i{}^{(2)} (t, \ \mathbf{x}), \quad \mathbf{x} \in S_{12} (t) \end{gathered} \tag{2.7}$$

where the superscripts denote that the respective quantities belong to the viscoelastic or plastic region.

The equations of state (1.1), (1.2) and (1.5), the equations of equilibrium (2.1), the relations between the deformation and displacement rates, the initial conditions (2.3), (2.4), the boundary conditions (2.5), (2.6), and the conditions at the interface constitute the complete system of equations and conditions of the boundary value problem of the theory of the viscoelastic-plastic state for a non-uniformly aging growing body.

Note that besides growing bodies, bodies undergoing a change in their shape owing to continuous removal of some part of their volume may be considered. As an example of this, we can mention damage to the elements of a structure by corrosion and cavitation, the action of other kinds of agressive media, ablation during blowing, burn-up of solid fuels, thawing, evaporation, etc. Situations are also possible in which the accretion and particle removal from bodies under load occur simultaneously at different sections of its boundary. The formation of the stress field can be studied using the general formulation of the boundary value problem for a body with changing boundary presented here, without any modification.

3. **The problem of the continuous growth of a hollow sphere.** 1°. *Statement of the problem.* Let the process of growth of a hollow sphere begin from the generation at the instant of time $t = 0$ of an elementary spherical layer of radius a. Then, (at $t > 0$) the sphere radius increases in accordance with the specified law $b = b (t)$. We will assume the function $b(t)$ to be monotonically increasing and continuous. At the instant t_0 $(b (t_0) = b_0)$ an internal pressure $p_0 > 0$ is applied to the sphere which subsequently varies in conformity with the law $p = p (t) > 0$ $(p (t_0) = p_0)$. It is assumed that up to the instant t_0 the growth of the sphere occurs without preliminary stressing of the layers. Suppose, furthermore, that the inner part of the sphere under the initially applied pressure passes into the plastic state. By the spherical symmetry of the problem, the boundary surface of the plastic zone is

a sphere of time-dependent radius $c(t)$.

For the function $\tau^*(r)$ that defines the instant of production (generation) of the elementary layer of the hollow sphere of radius r we have the relations $\tau^*(a) = 0$, $\tau^*(b_0) = t_0$, $b(\tau^*(r)) \equiv r$, $\tau^*(b(t)) \equiv t$. Hence the functions τ^* and b are reciprocal.

Let us write the relations defining the statement of the problem.

At any point of a hollow sphere of radius $(a \leqslant r \leqslant b(t))$ Eqs.(3.1) (given below) of quasi-static equilibrium must be satisfied (the subscripts r, θ denote the radial and peripheral component, respectively), the geometric relations (3.2) between the rates of deformation and displacement, Eq.(3.3) of the compatibility of the rates of deformation, and the defining equation (3.4) for the volume deformation have the form

$$\frac{\partial \sigma_r(t,r)}{\partial r} + \frac{2}{r}[\sigma_r(t,r) - \sigma_\theta(t,r)] = 0 \qquad (3.1)$$

$$\varepsilon_r^{\cdot}(t,r) = \frac{\partial u^{\cdot}(t,r)}{\partial r}, \qquad \varepsilon_\theta^{\cdot}(t,r) = \frac{u^{\cdot}(t,r)}{r} \qquad (3.2)$$

$$\frac{\partial \varepsilon_\theta^{\cdot}(t,r)}{\partial r} + \frac{1}{r}[\varepsilon_\theta^{\cdot}(t,r) - \varepsilon_r^{\cdot}(t,r)] = 0 \qquad (3.3)$$

$$\varepsilon(t,r) = \frac{\sigma(t,r)}{3K} \qquad (3.4)$$

$$\varepsilon = \frac{\varepsilon_r + 2\varepsilon_\theta}{3}, \qquad \sigma = \frac{\sigma_r + 2\sigma_\theta}{3}, \qquad K = \mathrm{const}$$

In the viscoelastic region $(c(t) \leqslant r \leqslant b(t))$ we have for the shear deformation the defining equation

$$e_{r(\theta)}(t,r) = \mathrm{L}(s_{r(\theta)}) \qquad (3.5)$$

$$\mathrm{L}(\alpha) = \frac{\alpha(t,r)}{2G(t-\tau^*(r))} - \int_{\tau^\circ(r)}^{t} \frac{\alpha(\tau,r)}{2G(\tau-\tau^*(r))} Q(t-\tau^*(r), \tau-\tau^*(r))\, d\tau$$

where $\tau^\circ(r)$ is the instant the stress is applied to the layer of radius r, and the boundary condition is

$$\sigma_r(t,\ b(t)) = 0 \qquad (3.6)$$

In the plastic region the material mechanical behaviour is defined by formulas similar to the relations of deformational plasticity

$$e^P_{r(\theta)}(t,r) = \frac{e_*^P(t,r)}{k(t-\tau^*(r))} s_{r(\theta)}(t,r) \qquad (3.7)$$

When $a \leqslant r \leqslant c(t)$ the condition of plasticity (k is the plastic limit for pure shear)

$$\sigma_\theta(t,\ r) - \sigma_r(t,\ r) = \sqrt{3}k(t-\tau^*(r)) \qquad (3.8)$$

is also satisfied.

On the inner surface of a hollow sphere we have the condition

$$\sigma_r(t,\ a) = -p(t) \qquad (3.9)$$

It is assumed, in accordance with the statement of the initial boundary value problem, that at the instant of accretion of an elementary layer of some radius r, its deformations and stresses are specified by

$$\varepsilon_{r(\theta)}(\tau^*(r),\ r) = \overset{\circ}{\varepsilon}_{r(\theta)}(r) \qquad (3.10)$$

$$\sigma_{r(\theta)}(\tau^*(r),\ r) = \overset{\circ}{\sigma}_{r(\theta)}(r), \quad b_0 \leqslant r \leqslant b(t)$$

where $\overset{\circ}{\varepsilon}_{r(\theta)}$, $\overset{\circ}{\sigma}_{r(\theta)}$ are continuous functions of the radius linked by relations of the form (2.4). As indicated in Sect.2, the boundary condition on the external surface of the growing hollow sphere, must be taken into account when establishing initial stresses. In this case, in view of condition (3.6) we, obviously, must have $\sigma_r^\circ(r) \equiv 0$.

Moreover, at the interface of the viscoelastic and plastic regions the conditions of continuity of the components of the stress and radial displacement are formulated as follows:

$$\sigma_r^{(1)}(t,\ c(t)) = \sigma_r^{(2)}(t,\ c(t)) \qquad (3.11)$$

$$\sigma_\theta^{(1)}(t,\ c(t)) = \sigma_\theta^{(2)}(t,\ c(t))$$

$$u^{(1)}(t,\ c(t)) = u^{(2)}(t,\ c(t))$$

At the external surface of the original hollow sphere the condition of continuity of the radial stress

$$\sigma_r^{(2)}(t,\ b_0) = \sigma_r^{(3)}(t,\ b_0) \qquad (3.12)$$

must be satisfied.

The superscripts 1, 2, 3 denote that the quantities belong to the following ranges of the radial component variation $a \leqslant r \leqslant c(t)$, $c(t) \leqslant r \leqslant b_0$, $b_0 \leqslant r \leqslant b(t)$.

2°. *Solution of the problem in the case of perfectly elastic volume deformation.* Substituting the condition of plasticity (3.8) into the equilibrium equation (3.1), and integrating with respect to the radius from a to r, taking the boundary condition (3.9) into account, we obtain for the radial stress in the plastic region the expression

$$\sigma_r^{(1)}(t,r) = -p(t) + 2\sqrt{3}\int_a^r \rho^{-1}k(t-\tau^*(\rho))\,d\rho \tag{3.13}$$

Let us determine the stresses in the viscoelastic region. For the components of total deformation when $c(t) \leqslant r \leqslant b(t)$ we have

$$\varepsilon_r = \frac{2}{3}\mathbf{L}(\sigma_r - \sigma_\theta) + \frac{\sigma}{3K} \tag{3.14}$$

$$\varepsilon_\theta = \frac{1}{3}\mathbf{L}(\sigma_\theta - \sigma_r) + \frac{\sigma}{3K}$$

The deviators of the stress and deformation intensities are defined as follows.

$$s_* = \frac{\sigma_\theta - \sigma_r}{\sqrt{3}}, \qquad e_* = \frac{\varepsilon_\theta - \varepsilon_r}{\sqrt{3}}$$

and by virtue of (3.5) these quantities are connected by the relation

$$e_* = \mathbf{L}(s_*) \tag{3.15}$$

Hence

$$\varepsilon_r = -\frac{2e_*}{\sqrt{3}} + \frac{\sigma_r + 2\sigma_\theta}{9K}$$

$$\varepsilon_\theta = \frac{e_*}{\sqrt{3}} + \frac{\sigma_r + 2\sigma_\theta}{9K}$$

Substituting these expressions into the equations of compatibility of the deformation rates (3.3) we obtain

$$\frac{\partial e_*^{\cdot}}{\partial r} + 3\frac{e_*^{\cdot}}{r} + \frac{1}{3K}\left(\sqrt{3}\,\frac{\partial \sigma_\theta^{\cdot}}{\partial r} - \frac{\partial s_*^{\cdot}}{\partial r}\right) = 0 \tag{3.16}$$

From the equilibrium equation we have

$$\frac{\partial \sigma_\theta^{\cdot}}{\partial r} - \sqrt{3}\left(\frac{\partial s_*^{\cdot}}{\partial r} + 2\frac{s_*^{\cdot}}{r}\right) = 0 \tag{3.17}$$

Eliminating $\partial\sigma_\theta^{\cdot}/\partial r$ from (3.16), (3.17) and integrating the equation obtained with respect to r and, then, with respect to t, taking the initial conditions (3.10) into account, we obtain

$$3Ke_*(t,r) + 2s_*(t,r) = \begin{cases} A(t)/r^3, & c(t) \leqslant r \leqslant b_0 \\ [A(t) - A(\tau^*(r))]/r^3 + 3KT(r), & b_0 \leqslant r \leqslant b(t) \end{cases} \tag{3.18}$$

$$T(r) = e_*(\tau^*(r), r) + \frac{2}{3K}s_*(\tau^*(r), r)$$

where A is a function of time, to be determined, and the radial coordinate function $T(r)$ defines the initial stress-strain state of the growing layer of radius r.

Let us consider regions 2 and 3 separately. Substituting (3.15) into the first relations of (3.18), we obtain the Volterra integral equation of the second kind in s_*.

$$\mathbf{L}(s_*) + \frac{2}{3K}s_*(t,r) = \frac{A(t)}{3Kr^3}$$

We write the transformation of this equation in the form

$$s_*(t,r) = \frac{E^\circ(t-\tau^*(r))}{3Kr^3}\mathbf{R}(A) \tag{3.19}$$

$$E^\circ(t-\tau^*(r)) \equiv \left\{\frac{1}{2G(t-\tau^*(r))} + \frac{2}{3K}\right\}^{-1}$$

where $\mathbf{R}$ is the linear integral operator acting on some function $\alpha(t,r)$ in accordance with the following definition:

$$\mathbf{R}(\alpha) \equiv \alpha(t,r) + \int_{\tau^\circ(r)}^{t} \alpha(\tau,r) R(t-\tau^*(r),\ \ \tau-\tau^*(r))\, d\tau \tag{3.20}$$

where $R(t,\ \tau)$ is the resolvent of the kernel $^1/_2E^\circ(\tau)\, Q\ (t,\ \tau)\ /\ G\ (\tau)$. Note that in region 2 $\tau^\circ(r) \equiv t_0$.

Substituting (3.19) into the equation of equilibrium and integrating with respect to the radius from some r to b_0, we obtain

$$\sigma_r^{(2)}(t,r) = \sigma_r(t,b_0) - \frac{2}{\sqrt{3}\,K}\Big\{A(t)\int_r^b \rho^{-4} E^\circ(t-\tau^*(\rho))\,d\rho + \int_t^t A(\tau)\Big[\int_r^{b_0} \rho^{-4}E^\circ(t-\tau^*(\rho))\,R(t-\tau^*(\rho),\tau-\tau^*(\rho))\,d\rho\Big]\,d\tau\Big\} \tag{3.21}$$

In (3.21) there appears the so far undetermined radial stress on the external surface of the original hollow sphere.

In a similar manner we can determine the radial stress in region 3 $(b_0 \leqslant r \leqslant b\ (t))$, i.e. in the accreted region. From (3.18), instead of (3.19) we have here

$$s_*(t,r) = E^\circ(t-\tau^*)\Big\{\frac{1}{3Kr^3}\mathbf{R}(A) + \Big[T(r) - \frac{A(\tau^*)}{3Kr^3}\Big]\mathbf{R}(1)\Big\} \tag{3.22}$$

where $\tau^* = \tau^*(r)$ and $\mathbf{R}$ is the operator (3.20) with $\tau^\circ \equiv \tau^*$.

Subsituting (3.22) into (3.1), integrating over the radius from a certain r to $b\ (t)$ taking boundary condition (3.6) into account, changing the order of integration with respect to ρ and τ, and making the change of variables $\xi = \tau^*(\rho)$ (or, which amounts to the same, $\rho = b\ (\xi)$), after some reduction we obtain

$$\begin{aligned}\sigma_r^{(3)}(t,r) = &-\frac{2}{\sqrt{3}\,K}\Big\{A(t)\int_{\tau^*(r)}^{t} \omega(t,\xi)\,d\xi - \\ &\int_{\tau^*(r)}^{t} A(\tau)\Big[\omega(t,\tau)Z(t,\tau) - \int_{\tau^*(r)}^{\tau}\omega(t,\xi)R(t-\xi,\tau-\xi)\,d\xi\Big]d\tau - \\ &2\sqrt{3}\int_{\tau^*(r)}^{t} T(b(\xi))E^\circ(t-\xi)Z(t,\xi)\lambda(\xi)\,d\xi\end{aligned} \tag{3.23}$$

$$\omega(t,\eta) \equiv E^\circ(t-\eta)\frac{b'(\eta)}{b^4(\eta)}, \qquad \lambda(\xi) \equiv \frac{b'(\xi)}{b^4(\xi)}$$

$$Z(t,\eta) \equiv 1 + \int_\eta^t R(t-\eta,\zeta-\eta)\,d\zeta$$

where b' is the derivative of the function b.

The expression for the circumferential stress $\sigma_\theta\ (t,\ r)$ is determined in region 1 from (3.8) and (3.13), in region 2 from (3.19) and (3.23), and in region 3 from (3.22) and (3.23).

Taking into account the continuity condition (3.12), we obtain from (3.23) an expression for the stress $\sigma_r\ (t,\ b_0)$ in terms of the function A. Substituting the expressions obtained for the stress components into the continuity conditions (the first two formulas (3.11)), we obtain

$$\begin{aligned}&A(t)\int_{\tau^*(c)}^{t}\omega(t,\xi)\,d\xi - \int_{t_0}^{t}A(\tau)\Big\{\omega(t,\tau)Z(t,\tau) - \\ &\int_{\tau^*(c)}^{\tau}\omega(t,\xi)R(t-\xi,\tau-\xi)\,d\xi\Big\}d\tau = \frac{\sqrt{3}}{2}Kp(t) - \\ &3K\Big\{\int_0^{\tau^*(c)} k(t-\xi)\lambda(\xi)\,d\xi + \int_{t_0}^{t}T(b(\xi))E^\circ(t-\xi)Z(t,\xi)\lambda(\xi)\,d\xi\Big\}\end{aligned} \tag{3.24}$$

$$A(t) + \int_{t_0}^{t}A(\tau)R(t-\tau^*(c),\tau-\tau^*(c))\,d\tau = 3Kc^3\,\frac{k(t-\tau^*(c))}{E^\circ(t-\tau^*(c))} \tag{3.25}$$

The radius of the plastic region corresponds to the instant t, i.e. $c = c\ (t)$. The solution of the problem has thus been reduced to finding the functions $A\ (t)$ and $c\ (t)$ from the system of integral equations (3.24) and (3.25) that are linear with respect to the function $A\ (t)$. After solving this system of equations, the deformations in the viscoelastic region

$c(t) \leqslant r \leqslant b(t)$ are determined using (3.14).

We will now determine the deformed state in the plastic region. For the deformation components we have

$$\varepsilon_r = \mathrm{L}(s_r) + \frac{e_*^P}{k} s_r + \frac{\sigma}{3K}$$

$$\varepsilon_\theta = \mathrm{L}(s_\theta) + \frac{e_*^P}{k} s_\theta + \frac{\sigma}{3K}$$

Since for $a \leqslant r \leqslant c(t)$ the condition of plasticity (3.8) applies, then $s_* = k$ and, consequently,

$$\varepsilon_r = -\frac{2}{\sqrt{3}}\{\mathrm{L}(k) + e_*^P\} + \frac{1}{9K}(\sigma_r + 2\sigma_\theta) \tag{3.26}$$

$$\varepsilon_\theta = \frac{1}{\sqrt{3}}\{\mathrm{L}(k) + e_*^P\} + \frac{1}{9K}(\sigma_r + 2\sigma_\theta)$$

From the compatibility equation of the deformation components we find

$$\frac{\partial}{\partial r}\{\mathrm{L}(k) + e_*^P\} + \frac{3}{r}\{\mathrm{L}(k) + e_*^P\} - \frac{1}{3K}\frac{\partial k}{\partial r} + \frac{1}{\sqrt{3}K}\frac{\partial \sigma_\theta}{\partial r} = 0 \tag{3.27}$$

and from the equilibrium equation we have

$$\frac{\partial \sigma_\theta}{\partial r} = \sqrt{3}\left(\frac{\partial k}{\partial r} + 2\frac{k}{r}\right) \tag{3.28}$$

Substituting (3.28) into (3.27) and integrating with respect to r, we obtain

$$e_*^P(t, r) = \frac{B(t)}{r^3} - \frac{2}{3K}k - \mathrm{L}(k)$$

where B is an undetermined function of time.

Since $u = r\varepsilon_\theta$ when $a \leqslant r \leqslant b_0$, the conditions of displacement continuity (the third relation of (3.11)) when $r = c(t)$ must have $e_\theta^P(t, c) = 0$. It then follows from (3.7) that $e_*^P(t, c) = 0$, and for the function B we have the expression

$$B(t) = c^3\left\{\mathrm{L}(k)|_{r=c} + \frac{2}{3K}k(t - \tau^*(c))\right\}, \qquad c = c(t)$$

The deformation components in the plastic region can now be determined using (3.26).

Note that the solution of the problem, determined by the system of Eqs.(3.24) and (3.25) is real when the conditions $c^\cdot \geqslant 0$ and $c \leqslant b_0$ are satisfied when $t \geqslant t_0$. If the radius of the plastic region at some instant of time reaches the external radius of the original hollow sphere, the subsequent analysis necessitates some modification of relations that determine the solution of this problem. The wall thickness of the hollow sphere will then consist of two regions: a region of plastic state of material and a region of accretion. This case is considered similarly.

The elastic-plastic deformation of an inhomogeneous hollow sphere of elastically compressible material was investigated in /8/. The problem of plastic wave propagation in a thick-walled pipe of viscoelastic-plastic material was considered in /9–11/. The solution of the problem of the growth of an incompressible viscoelastic hollow sphere under conditions of non-uniform aging was given in /12/.

3°. *Solution of the problem in the case of an incompressible material.* Let us consider the above problem with the additional assumption that the material is incompressible.

Let us assume that the prestressing of layers at $r \geqslant b_0$ is determined by the equations

$$\sigma_r(\tau^*(r),\ r) = 0, \quad \sigma_\theta(\tau^*(r),\ r) = \sigma_\theta^0(r)$$

where $(\sigma_\theta^0(r)$ is a given function.

From the condition of incompressibility written in terms of velocities

$$\varepsilon_r^\cdot(t,\ r) + 2\varepsilon_\theta^\cdot(t,\ r) = 0$$

and relations (3.2), taking prestressing into account, we obtain

$$\varepsilon_\theta(t, r) = -\frac{1}{2}\varepsilon_r(t, r) = \begin{cases} A(t)/r^3, & a \leqslant r \leqslant b_0 \\ [A(t) - A(\tau^*(r))]/r^3 + {}^1/_2\varepsilon^\circ(r), & b_0 \leqslant r \leqslant b(t) \end{cases} \tag{3.29}$$

$$\varepsilon^\circ(r) = \sigma_\theta^\circ(r)/G(0)$$

Taking into account relations (3.29), from the defining equation (3.5) we obtain for the radial stress in regions 2 and 3 the expressions

$$\sigma_r^{(2)}(t,r)=\sigma_r(t,b_0)-12\left\{A(t)\int\limits_{\tau^*(r)}^{t_0} g(t,\xi)\,d\xi+\int\limits_{t_0}^{t}A(\tau)\left[\int\limits_{\tau^*(r)}^{t_0} g(t,\xi)\,R(t-\xi,\tau-\xi)\,d\xi\right]d\tau\right\} \tag{3.30}$$

$$\sigma_r^{(3)}(t,r)=-12\left\{A(t)\int\limits_{\tau^*(r)}^{t} g(t,\xi)\,d\xi-\right. \tag{3.31}$$

$$\int\limits_{\tau^*(r)}^{t}A(\tau)\left[g(t,\tau)Z(t,\tau)-\int\limits_{\tau^*(r)}^{\tau} g(t,\xi)\,R(t-\xi,\tau-\xi)\,d\xi\right]d\tau+$$

$$\left.\frac{1}{6}\int\limits_{\tau^*(r)}^{t}\varepsilon^\circ(b(\xi))\,G(t-\xi)\,Z(t,\xi)\,\lambda(\xi)\,d\xi\right\}$$

$$g(t,\eta)\equiv G(t-\eta)\,b'(\eta)\,[b(\eta)]^{-4}$$

The resolving system of equations in this case has the form (the value of c corresponds to the instant t)

$$A(t)\int\limits_{\tau^*(c)}^{t} g(t,\xi)\,d\xi-\int\limits_{t_0}^{t}A(\tau)\left\{g(t,\tau)Z(t,\tau)-\int\limits_{\tau^*(c)}^{\tau} g(t,\xi)\,R(t-\xi,\tau-\xi)\,d\xi\right\}d\tau=\frac{1}{12}\,p(t)- \tag{3.32}$$

$$\frac{1}{6}\left\{\sqrt{3}\int\limits_{0}^{\tau^*(c)}k(t-\xi)\,\lambda(\xi)\,d\xi+\int\limits_{t_0}^{t}\varepsilon^\circ(b(\xi))\,G(t-\xi)\times Z(t,\xi)\,\lambda(\xi)\,d\xi\right\}$$

$$A(t)+\int\limits_{t_0}^{t}A(\tau)\,R(t-\tau^*(c),\tau-\tau^*(c))\,d\tau=\frac{c^3}{2\sqrt{3}}\,\frac{k(t-\tau^*(c))}{G(t-\tau^*(c))} \tag{3.33}$$

4°. *Discussion of the results.* When analyzing the solution of the problem of the stress-strain state of a growing incompressible hollow sphere for the rheological characteristics of the material, the law of accretion and the acting load, we take the following relations:

$$G(t)=G_0,\ R(t,\tau)=\frac{\partial}{\partial\tau}\{(C_1+C_2e^{-\mu\tau})\,[1-e^{-\gamma(t-\tau)}]\}$$

$$k(t)=k_\infty\,(1-q_0e^{-\varkappa t}),\ b(t)=ae^{\beta t}$$

$$p(t)=p_0+p_1t,\quad \sigma_\theta^0(r)\equiv\Theta$$

where $G_0, C_1, C_2, k_\infty, q_0, p_0, p_1, \Theta, \mu, \gamma, \varkappa, \beta$ are positive constants. In this case all quadratures appearing in (3.30), (3.31) and in (3.32), (3.33) can be expressed in terms of elementary functions.

We use the following units: for length, the internal radius a of the hollow sphere, for time, the quantity $\varkappa^{-1}$, and for pressure, the limit value of the yield point k_∞. Retaining the previous notation for dimensionless quantities, we take the following numerical values for them: $b_0=2$; $G_0=100$; $p_0=1{,}65$; $C_1=0.05$; $C_2=0{,}75$; $q_0=0{,}5$; $\mu=1$; $\gamma=2$; $p_1=0.25$. In the calculations the parameters β, Θ were varied.

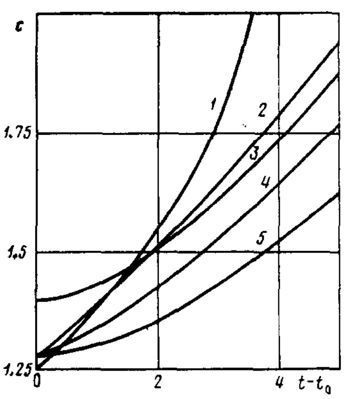

Fig.2

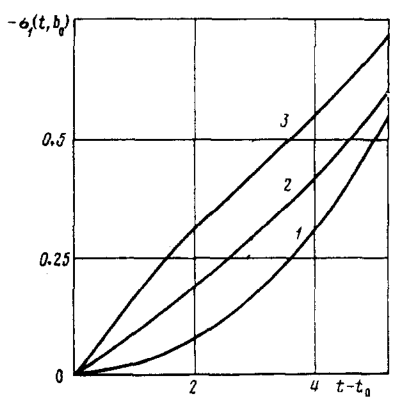

Fig.3

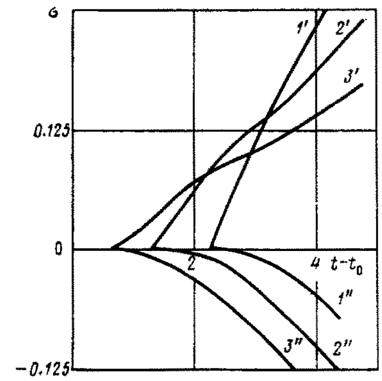

Fig.4

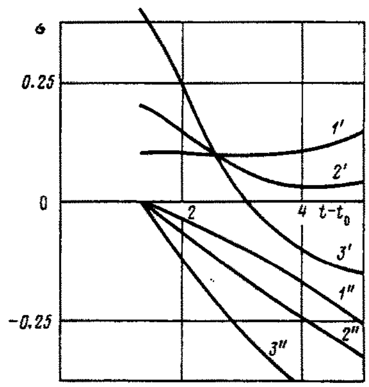

Fig.5

An analysis of the results of a numerical investigation of the solution of the problem shows that for the same rheological properties of the material and the law of internal pressure variation, the rate of growth and the degree of prestressing of the accreted layers have an important effect on the stress-strain state of the hollow sphere.

The calculated dependence of the plastic region radius on time is shown in Fig.2. Curves *1, 2, 3* are calculated for various rates of growth (the parameter β takes the values 0.025; 0.175; 0.35) respectively, without prestressing the layers ($\theta = 0$)). It can be seen that the initial value of the radius of the plastic region is larger, the larger the accretion rate. This explained by noting that at a relatively high rate of growth, the yield stress in the original hollow sphere at the instant the inner pressure is applied is comparatively far from its limit value, while when the rate of growth is low, the material has time to "age" considerably and become more rigid. However, subsequently for a high rate of growth the plastic region extension is slowed down, since in this case the redistribution of the stresses from elements of the original sphere to the newly created layers is more intensive. Curves *4* and *5* are calculated for the same ates of growth ($\beta = 0.175$) at different degrees of prestressing of accreted layers (the parameter θ is respectively equal to 0.2 and 0.4)). It follows from a comparison of curves *2, 4, 5* that an increase in the degree of prestressing of accreted layers at an equal rate of extension of the hollow sphere external boundary, results in a slowing down of the expansion of the plastic region.

The effect of the degree of prestressing of the layer in the growth region on the time of radial stress at the boundary of the original hollow sphere $\sigma_r(t, b_0)$ with the same law of growth ($\beta = 0.175$) is shown in Fig.3. Curves *1, 2,* and *3* correspond to the value of the prestressing parameter $\theta = 0$; 0.2; 0.4. Note that strengthening of prestressing in accretion is equivalent to the application to the original hollow sphere of some supplementary external compression . This reduces the circumferential stress in the viscoelastic region of the original hollow sphere, and as previously noted, restrains the expansion of the plastic region.

The effect of the growth rate (when there is no prestressing) on the stress state of the layer $r = 2.5$ located in the accreted region is illustrated in Fig.4. Curves *1, 2, 3* correspond to three values of the growth rate $\beta = 0.1; 0.175; 0.35$. A prime on the number denotes curves that correspond to circumferential stress and numbers with two primes correspond to the radial stress. It can be seen that the increase in growth rate results in an increase in the radial stress in layer considered. For instance, at the instant $t = 3$ the absolute magnitude of the radial stress when $\beta = 0.35$ (curve *3''*) exceeds the value at $\beta = 0.1$ (curve *1''*) by a factor of 10.5. Simultaneously the increase in growth note is accompanied by a drop in the rate of increase of the circumferential stress. This is explained by the increasing compressive reaction of the accreted layers.

The effect of prestressing on the magnitude of the stresses in the same layer $r = 2.5$ for the same law of growth ($\beta = 0.175$) is shown in Fig.5. Curves *1, 2, 3* correspond to three values of the prestress parameter $\theta = 0.1; 0.2; 0.4$. As seen from Fig.5, prestressing may lead to a qualitative change in the nature of the dependence of the circular stress in the accreted region on time. When the initial tension in the accreted layers is fairly intensive, the circumferential stress changes its sign, which indicates a loss of prestressing in some internal part of the wall of the accreted hollow sphere.

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THE DEFORMATION THEORY OF PLASTICITY OF ANISOTROPIC MEDIA*

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Mutually inverse defining equations of the deformation theory of plasticity of media with arbitrary anisotropy are written assuming the relations between the stresses and deformations to be quasilinear. The conditions of plasticity and unloading are considered. Theorems are proved on the existence and uniqueness of solutions of the quasistatic problem of the deformation theory of plasticity and of simple loading. The method of successive approximations for solving the problem is considered, and its convergence is proved. Various means of simplyfying the theory are considered. Theorems of minimum Lagrangian and the maximum of the Castiglianian are proved.

In the deformation theory of plasticity the stresses and deformations are connected by finite relations. When these relations are quasilinear (tensor-linear) /1/, and the medium is isotropic, for simple processes /2/ all theories of plasticity agree with the deformation theory (the theory of small elastic-plastic deformations) /3/. However, in practice that theory is used for a wider class of processes of deformations. The advantage of this theory is its simplicity, the mutually inverse relations between the stresses and deformations, the availability of theorems of existence and uniqueness and of the minimum of the Lagrangian and maximum of the Castiglianian, of the theorem of simple loading and unloading /2/, and also the existence of an effective method of solving quasistatic problems, the method of elastic solutions /2/, whose convergence was adequatly analyzed in /4, 5/. Below a deformation theory is constructed for initially anisotropic media.

1. Let the symmetric stress tensor $\boldsymbol{\sigma}$ be a tensor function of the small deformation tensor $\boldsymbol{\varepsilon}$; this function is invariant to transformations that characterize certain classes of anisotropy. The function can be represented in the form of the dependence of the tensor $\boldsymbol{\varepsilon}$ and some "parametric" tensors $\mathbf{A}_1, \mathbf{A}_2, \ldots$ that define the considered anisotropy class /1/. Let us assume that this anisotropic function is quasilinear (tensor-linear) /1, 6/. This means that its polynomial representation /7/ contains only tensors linearly dependents on $\boldsymbol{\varepsilon}$,

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